Spectral noncommutative geometry and quantization: a simple example

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We explore the relation between noncommutative geometry, in the spectral triple formulation, and quantum mechanics. To this aim, we consider a dynamical theory of a noncommutative geometry defined by a spectral triple, and study its quantization. In particular, we consider a simple model based on a finite dimensional spectral triple (A, H, D), which mimics certain aspects of the spectral formulation of general relativity. We find the physical phase space, Γ, which is the space of the onshell Dirac operators compatible with A and H. We define a natural symplectic structure over Γ and construct the corresponding quantum theory using a covariant canonical quantization approach. We show that the Connes distance between certain two states over the algebra A (two "spacetime points"), which is an arbitrary positive number in the classical noncommutative geometry, turns out to be discrete in the quantum theory, and we compute its spectrum. The quantum states of the noncommutative geometry form a Hilbert space K. D is promoted to an operator $\hat{D}$ on $\mathcal{H} = H \otimes K$. The triple $(A, \mathcal{H}, \hat{D})$ can be viewed as the quantization of the family of the triples (A, H, D).

I. WHY QUANTIZING A NONCOMMUTATIVE GEOMETRY

The idea that the geometric structure of physical spacetime could be noncommutative exists in different versions. In some of versions, the noncommutativity of geometry is viewed as a direct effect of quantum mechanics, which disappears in the limit in which we consider processes involving actions much larger than the Planck constant [1]. In the noncommutative geometry approach of Connes et. al. (NCG) [2–5], on the other hand, noncommutativity is introduced as a feature of spacetime which exists independently from quantum mechanics. For instance, in the noncommutative version of the standard model [3], the theory is defined over a noncommutative spacetime, and is *then* quantized along conventional perturbative lines. More ambitiously, the spectral triple formulation [4] includes the gravitational field as well. For the gravitational field, however, conventional perturbative quantization methods fail [6]. The problem of going from the noncommutative, but non-quantum-mechanical, spectral triple dynamics to the full quantum dynamics is thus open.

In a quantum theory that includes gravity, the geometric structure of spacetime is to be treated quantum mechanically. Therefore the noncommutative geometry of spacetime must be reinterpreted in quantum terms. Thus, in a theory of the physical world based on NCG and including quantum mechanics, the geometry of spacetime should be represented by a *quantization* of a noncommutative geometry.

There should therefore be two distinct sources of noncommutativity in the theory: the noncommutativity of the elements of the algebra describing spacetime *and*, separately, the noncommutativity of the quantum mechanical variables. In this note, we address the problem of understanding what a quantization of a noncommutative geometry might be, and what is the relation between geometric noncommutativity and quantum noncommutativity. We study this problem using a simple model derived from [5] with the aim of developing structures and notions which, hopefully, could guide us in addressing the same issue in a full model including general relativity.

In particular, we consider the spectral triple approach given in [4]. Within this approach, a dynamical model is given by the spectral triple (A, H, D), where H is a Hilbert space and A is a C^* algebra represented on H, which are fixed once and for all; while D is a Dirac operator (in the sense of [2]) in H, which codes the value of the dynamical fields, and in particular of the spacetime metric, that is, the gravitational field. Thus D is the dynamical variable of the model and represents a classical configuration of the theory. The dynamics is then given by an action $S[D]$. To quantize the theory, we must find the Hilbert space K of its quantum states. A state in K will represent a quantum state of the noncommutative geometry: roughly, a probabilistic quantum superpositions of (noncommutative) geometries. Such a state will assign not a number, but rather a probability distribution, to the observable distance $d(p, p')$ between any two points. Observable quantities will be represented by operators on K.

We construct the Hilbert space K and the dynamical operators for our simple model. From the quantum theory we obtain a concrete result: the physical distance between (certain) two points of the model, which in the classical theory can be an arbitrary nonnegative number d, turns out to be quantized as

$$d = \frac{L_P}{\sqrt{2n+1}}, \qquad n = 0, 1, 2, 3 \ldots \tag{1}$$

where L_P (the "Planck length") is the length determined

by $\hbar$ and the coupling constant of the theory. Furthermore, we show that the quantum theory can be compactly represented in terms of a novel triple $(A, \mathcal{H}, \hat{D})$, where $\mathcal{H}$ is the tensor product of H with the space of the quantum geometries K.

There exist several attempts to explore the relation between noncommutative geometry and quantum theory by studying quantum fields defined over a noncommutative geometry [1,8]. These attempts should not be confused with the present work. Here we are not concerned with the effect of a noncommutative structure of spacetime over quantum fields: we are concerned with the quantum mechanical properties of the noncommutative geometry itself.

II. PRELIMINARIES

A. Mechanics without preferred time

We begin by recalling a few simple points about classical and quantum mechanics. This will fix notation and provide an appropriate conceptual framework. We need to choose a language sufficiently general to deal with theories, such as general relativity (GR), in which time evolution enters in a non-simple manner.

A dynamical theory is defined by a set of equations, the equations of motion, for a set of variables, the dynamical variables. By dynamical variable we mean here the full "history", or "motion", of the physical system. We denote the space of these variables as $\mathcal{C}$, or extended configuration space. For instance, the dynamics of an oscillator is given by the equation of motion $d^2x(t)/dt^2 = -\omega x(t)$, where the variable is the *function* $x : t \mapsto x(t)$. Thus, in this example the extended configuration space $\mathcal{C}$ is $C_\infty(R)$. When time evolution is standard, as in this example, $\mathcal{C}$ is the space of the maps from R (the time) into a configuration space. However, we are interested also in systems in which $\mathcal{C}$ does not have this form. We consider Lagrangian systems, in which the equations of motion can be expressed as the vanishing of the first variation of a function $S[x]$ on $\mathcal{C}$. $S[x]$ is the action functional.

The solutions of the equations of motion form a subspace of $\mathcal{C}$. This subspace is the phase space of the system, and we denote it Γ. In the oscillator, the solutions of the equations of motion are $x(t) = A\sin(\omega t+\phi)$. Thus, Γ is a two-dimensional subspace of $\mathcal{C}$ coordinatized by A and ϕ.*

A point s in Γ represents a physically realizable (that is, compatible with the classical equations of motion), or "onshell", history of the system. This is a classical state of the system. Here, "state" is used in the same (atemporal) sense as "Heisenberg state" in quantum theory. An observable quantity f corresponds to a function $f : s \in \Gamma \mapsto f(s) \in R$, where $f(s)$ is the predicted value of f in the state s.† The phase space carries a symplectic structure. In the example, $\Omega = dp \wedge dq = \omega A\, dA \wedge d\phi$. This defines Poisson brackets between observables. If the system has a standard time structure, the symplectic structure can be determined by standard methods. The definition of Ω in more general cases can be problematic. The Poisson bracket structures of the observables is the starting point for their quantization.

B. The spectral formulation of general relativity

The model we introduce in the next section is meant to mimic some of the features of the spectral formulation of GR [4,7]. We briefly recall this formulation. Consider euclidean GR over a fixed compact‡ four dimensional manifold M. Let $\mathcal{G}_M$ be the space of the riemanian metrics $g : M \times M \to R^+$ over M. In the standard formulation of the theory, $\mathcal{G}_M$ is taken as the extended configuration space

$$\mathcal{C} \equiv \mathcal{G}_M. \tag{2}$$

This is the space of the (four-dimensional) gravitational fields. The action is then chosen to be the the well known Einstein-Hilbert action

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* In the example of the oscillator, and anytime time evolution is standard, we can represent Γ as a space of initial data. In fact, each set of initial data determines a solution and viceversa. To do this, we have to fix a time, say $t = t_0$. Define then $q = x(t_0)$ and $p = dx(t)/dt|_{t=t_0}$. The points in Γ can then be coordinatized by (p, q) instead of (A, ϕ). The relation between the two set of coordinates on Γ is immediately obtained from the definitions as $q = A\sin(\omega t_0 + \phi)$ and $p = \omega A\cos(\omega t_0 + \phi)$. The (p, q) coordinatization of Γ is the one commonly introduced in textbooks. The definition of the phase space as the space of the solutions of the equations of motion (known since Lagrange) has the advantages of being more covariant (this becomes clear in field theory), of not requiring the fixing of a preferred time t_0 and of being extendible to the dynamical systems without standard time evolution that concern us here.

† For instance, the position x_t of the oscillator at time t is the function $x_t(A, \phi) = A\sin(\omega t + \phi)$ on Γ.

‡ Euclidean GR over a compact manifold is a theory which is likely to admit only a finite dimensional space of solutions, and whose relation with physical GR is questionable. However, the theory is, by itself, relevant as a model of diffeomorphism invariant field theory of the geometry. More importantly, the experience with conventional euclidean quantum field theory suggests that the quantization of the euclidean theory with compact time might be of great relevance for understanding the true quantum theory even if the corresponding classical theories have extremely different properties.

$$S[g]=\int d^4x\ \sqrt{g}R \tag{3}$$

The phase space Γ is the space of the Einstein spaces, namely the gravitational fields that solve the Einstein equations.

The theory can be formulated also as follows. Fix the 4d compact (spin) manifold M. Consider the spectral triples (A,H,D). Here $A=C_\infty(M)$; the Hilbert space H is the (Hilbert completion of the) space of the half-densitized Dirac spinor fields on M. Notice that H is defined also in the absence of a Riemanian structure[§]. H carries a representation π of A, where $\pi(a)$ is the operator that multiplies the spinor by the function $a\in A$. A and H are fixed structures. We then consider the space $\mathcal{D}_{(A,H)}$ of all the Dirac operators D, in the sense of Connes [2], namely the set of the operators D on H such that (A,H,D) is a spectral triple. We take the space of the Dirac operators as the extended configuration space

$$\mathcal{C}\equiv\mathcal{D}_{(A,H)}. \tag{4}$$

Thus, a Dirac operator represents here a history of the gravitational field. The action is chosen to be

$$S[D]=Tr\ [f(D)] \tag{5}$$

where $f(\cdot)$ is a suitable simple function given in [4,7]. The dynamical system defined by (4) and (5) is physically equivalent to the one defined by (2) and (3) (see [4,7] for qualifications). The reason is that given a metric $g\in\mathcal{G}_M$, there exists a corresponding Dirac operator $D(g)$ on H, defined with standard techniques. Viceversa, given an operator D in $\mathcal{D}_{(A,H)}$ there is a Riemanian metric g on M defined by $g=d$, where d is given by the NCG distance formula

$$d(p,p')=sup_{\{a\in A,|[D,\pi(a)]|\leq 1\}}\,|p(a)-p'(a)|,\quad p,p'\in M, \tag{6}$$

which is such that $D=D(g)$. In (6), $p(a)=a(p)$ and $p'(a)=a(p')$ are the Gel'fand states (the points of the space M correspond to states over the algebra $C_\infty(M)$). Thus, there is a one-to-one correspondence between the space $\mathcal{D}_{(A,H)}$ of the Dirac operators and the space $\mathcal{G}_M$ of the Riemanian metrics, and we can identify the two spaces. Finally, it is shown in [2] that $S[D(g)]\sim S[g]$ (again, see [4,7] for qualifications).

The key lesson we learn from this is that the space of the Dirac operators $\mathcal{D}_{(A,H)}$ can be viewed as the space $\mathcal{C}$ of the extended configurations of the gravitational field. Accordingly, the phase space Γ of the theory is the subspace of $\mathcal{D}_{(A,H)}$ where the first variation of the action functional $S[D]$ vanishes. In a (nonperturbative) quantization of GR, one chooses a coordinatization $f,f',\ldots$ of the phase space Γ of the theory and then searches for a representation of the Poisson algebra of the observables $f,f',\ldots$ in terms of self-adjoint operators on a Hilbert space K. If we want to explore non-perturbatively the quantum mechanics of a NCG model, we have thus to quantize a phase space Γ formed by the Dirac operator obeying appropriate equations of motion. In other words, we have to coordinatize this space, and represent the coordinates as operators on the quantum state space. Below, we complete this procedure for a simple, finite dimensional, model.

III. DEFINITION OF THE MODEL

We choose a finite dimensional spectral triple (A,H,D) (see [5]). Let A be the algebra $M_2\oplus C$, where M_2 is the algebra of complex 2×2 matrices and C is the complex plane. We write $a=(\mathrm{A},\alpha)\in A=M_2\oplus C$. Let H be the Hilbert space M_3, the linear space of complex 3×3 matrices Ψ with the scalar product $(\Psi,\Phi)=Tr[\Psi^\dagger\Phi]$, and let A act on H in the representation π given by

$$\pi(a)\Psi=\Psi_a\Psi \tag{7}$$

where

$$\Psi_a=\begin{pmatrix}\mathrm{A} & 0\\ 0 & \alpha\end{pmatrix}. \tag{8}$$

A Dirac operator on H [2,5], has the form (see [5])

$$D\Psi=\mathrm{D}\Psi+\Psi\mathrm{D}^\dagger \tag{9}$$

where

$$\mathrm{D}=\begin{pmatrix}0 & m\\ \overline{m} & 0\end{pmatrix}=\begin{pmatrix}0 & 0 & m_1\\ 0 & 0 & m_2\\ \overline{m}_1 & \overline{m}_2 & 0\end{pmatrix}. \tag{10}$$

m_1 and m_2 are complex numbers. As mentioned above, we view A and H as fixed structures, while D is the dynamical variable of the model. Since the Dirac operator (10) is determined by the two complex numbers $m_i, i=1,2$, the space $\mathcal{D}_{(M_2\oplus C, C^3)}$ of the Dirac operators compatible with the given A and H, namely the space of the dynamical variables, is isomorphic to C^2 and coordinatized by m_i. Thus the extended configurations space is

$$\mathcal{C}=\mathcal{D}_{(M_2\oplus C,\,M_3)}\sim C^2. \tag{11}$$

This is the analog, in our model, of the space of the configurations of the gravitational field in GR

We complete the definition of the spectral triple (see [2]) by defining

$$\begin{aligned}&J\Psi\equiv\Psi^\dagger;\\ &\Psi_\gamma\equiv\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\\ &\gamma\Psi\equiv\Psi_\gamma\Psi,\\ &\chi\Psi\equiv\gamma J\gamma J\Psi=\Psi_\gamma\Psi\Psi_\gamma.\end{aligned} \tag{12}$$

[§] I thank Alain Connes for clarifying this point.

The two involutions J ("charge conjugation") and χ ("parity") satisfy the properties needed to complete the definition of the spectral triple.

The dynamics is determined by choosing an action $S[D]$, namely a function on the space of the dynamical variables $\mathcal{C}$. Here we disregard the "spectral principle" [4] which requires the action to depend on the spectrum of D only: we leave the extension of the present ideas to genuinely spectral invariant actions to further developments. The simplest possibility is to have a "free" theory with a quadratic action. Thus we search an action of the form

$$S[D] = \frac{1}{2}\, Tr[\,\mathrm{D}\,\tilde{M}\,\mathrm{D}\,], \tag{13}$$

where $\tilde{M}$ is a 3×3 matrix that determines the equations of motion. It is easy to see that, because of the special form (10) of D, the action $S[D]$ in (13) can be rewritten as

$$S[D] = \overline{m}_i M_{ij} m_j, \tag{14}$$

where M is a 2×2 matrix. We want the action to be real, and thus M must be hermitian. We want the theory to have a nontrivial space of solutions, and thus M must have vanishing determinant. These requirements fix M up to a single complex parameter α and an overall normalization G that does not affect the field equations but is needed to set the right physical dimensions

$$M = \frac{1}{G}\begin{pmatrix} |\alpha|^2 & \overline{\alpha} \\ \alpha & 1 \end{pmatrix}. \tag{15}$$

(We write G in the denominator, following the GR use.) For simplicity, we further choose α to be a pure phase $\alpha = e^{i\phi}$, although this is not really needed for what follows. Thus

$$M = \frac{1}{G}\begin{pmatrix} 1 & e^{-i\phi} \\ e^{i\phi} & 1 \end{pmatrix}. \tag{16}$$

This completes the definition of the model. Let us now analyze its content.

By extremizing the action with respect to m_i and $\bar{m}_i$ we obtain the equations of motion

$$m_2 = e^{i\phi}\, m_1, \tag{17}$$

the "Einstein equations" of the model. These select, out of all the Dirac operators, a physical phase space Γ of physical ones (in GR: the Dirac operators corresponding to Einstein metrics). We say that D is "on shell" if it satisfies the equations of motion.

The phase space Γ is coordinatized just by m_1. Therefore Γ is isomorphic to the complex plane C. From now on we will write $m \equiv m_1$ for simplicity. The complex plane has a natural symplectic structure.

$$\Omega = \frac{i}{G}\, dm \wedge d\overline{m} = \frac{2}{G}\, d(\Re(m)) \wedge d(\Im(m)). \tag{18}$$

(G adjusts dimensions.) We take this as the physical symplectic structure of the system**. Thus the basic Poisson brackets are

$$\{m, \overline{m}\} = iG \tag{19}$$

A straightforward computation shows that the symplectic form Ω can be written directly in terms of the Dirac operator. We define by $\Omega_{ex} \equiv d\theta$, where

$$\theta \equiv \frac{i}{12G}\ \mathrm{Tr}[\gamma D dD] = \frac{i}{4G}\ Tr[\Psi_\gamma \mathrm{DdD}]. \tag{20}$$

(The first trace is the trace of the operators on H, the second is the trace of the 3×3 matrices.) Then Ω is the restriction on shell of Ω_{ex}.

The physical interpretation of D is that it determines the metric structure, and therefore the "gravitational field", over the space of the states over A. A state p over A is determined by a vector Ψ_p in H

$$p(a) = \langle \Psi_p | \pi(a) | \Psi_p \rangle. \tag{21}$$

In particular, consider the two vectors $\Psi_p = \psi_p \otimes \psi_p$ and $\Psi_{p'} = \psi_{p'} \otimes \psi_{p'}$, where $\psi_p = (0,0,1)$ and $\psi_{p'} = \frac{1}{\sqrt{2}}(e^{i\phi}, 1, 0)$. They define the states p and p' over A. Their distance is given by (6) and can be explicitly computed using Eq. (2.175) of Sec 2.4.1 in [5], obtaining

$$d(p, p') = \frac{1}{\sqrt{|m_1|^2 + |m_2|^2}} = \frac{1}{\sqrt{2}|m|}. \tag{22}$$

IV. QUANTIZATION

We want to promote m and $\overline{m}$ to operators $\hat{m}$ and $\hat{\overline{m}}$ on a Hilbert space K in such a way that their algebra represents ($i\hbar$ times) the Poisson algebra (19) and that the operators $\Re\hat{m} = 1/2(\hat{m} + \hat{\overline{m}})$ and $\Im\hat{m} = i/2(\hat{m} - \hat{\overline{m}})$, which correspond to real quantities, be self-adjoint. The solution is well known. The complex structure of Γ provides us with a preferred polarization: we choose a representation in which the wave functions depend on m and are independent from $\overline{m}$. Vectors in K are thus analytic functions $\psi(m)$ and

$$\begin{aligned} \hat{m}\ \psi(m) &= m\ \psi(m), \\ \hat{\overline{m}}\ \psi(m) &= \hbar G\ \frac{d}{dm}\psi(m). \end{aligned} \tag{23}$$

The scalar product that implements the desired reality conditions is

$$(\psi, \psi') = \frac{1}{\pi\hbar G}\int_C dm\ e^{-\frac{|m|^2}{\hbar G}}\ \overline{\psi(m)}\, \psi'(m) \tag{24}$$

**I thank Abhay Ashtekar for this suggestion

(The notation is $dm = d\Re(m)d\Im(m)$.) A convenient orthonormal basis is given by the monomials

$$\psi_n(m) = \frac{m^n}{\sqrt{n!}(\hbar G)^{n/2}} = \langle m|n\rangle, \tag{25}$$

where we have introduced the Dirac notation $|n\rangle$ for the basis elements. In this basis $\hat{m}$ and $\hat{\overline{m}}$,

$$\begin{aligned} \hat{m}\ |n\rangle &= \sqrt{\hbar G}\ \sqrt{n+1}\ |n+1\rangle, \\ \hat{\overline{m}}\ |n\rangle &= \sqrt{\hbar G}\ \sqrt{n}\ |n-1\rangle, \end{aligned} \tag{26}$$

are immediately recognized as the creation and annihilation operators. In fact, equations (23,24,25) form the standard definition of the harmonic oscillator quantum mechanics in the Bargmann representation. We write the vectors in K as

$$|\psi\rangle = \sum_n\ \psi_n\, |n\rangle, \tag{27}$$

namely we represent K as the l_2 space of sequences ψ_n.

The operator corresponding to the m_2 variable is related to $\hat{m}_1 = \hat{m}$ by the (Heisenberg) equations of motion

$$\hat{m}_2 = e^{i\phi}\ \hat{m}_1. \tag{28}$$

Consider now the distance d between the two points p and p'. In the classical theory, this is given by (22). The corresponding operator $\hat{d}$ in the quantum theory is obtained by replacing the classical quantities $m_i, \overline{m}_i$ with their quantum counterparts $\hat{m}_i, \hat{\overline{m}}_i$. Using (28), we obtain

$$\hat{d} = \frac{1}{\sqrt{2N}}, \tag{29}$$

where N is the operator corresponding to the classical quantity $m\overline{m}$. We encounter here an ordering ambiguity. We compute easily from (26)

$$N|n\rangle = \hbar G\ (n+c)\ |n\rangle. \tag{30}$$

where we have $c = 0$ if we order N as $N = \hat{m}\hat{\overline{m}}$; we have $c = 1$ with the inverse ordering $N = \hat{\overline{m}}\hat{m}$; and we have the well known "harmonic oscillator vacuum energy" $c = 1/2$ with the "natural" symmetric ordering, which we assume from now on.

Thus we obtain the result that in the quantum theory the distance $d(p, p')$ is quantized, with discrete eigenvalues

$$d(p, p') = \frac{1}{\sqrt{(2n+1)\hbar G}} = \frac{L_P}{\sqrt{2n+1}}, \tag{31}$$

where

$$L_P = \frac{1}{\sqrt{\hbar G}}. \tag{32}$$

In the classical limit in which $\hbar$ is small and the quantum number n is large, the eigenvalues of the distance become more and more dense and approximate the classical continuum.

V. QUANTUM NONCOMMUTATIVE GEOMETRY

So far we have introduced two Hilbert spaces. The first is the Hilbert space of the spectral triple we started from. This is $H = M_3$, with vectors Ψ. We write their components also as $\psi^{ab}, a, b = 1, 2, 3$. H is the space of the "fermions" over the spectral triple. The second is the Hilbert space of the quantum theory $K = l_2$, with vectors $\psi_n, n = 0, 1, 2, \ldots$. K is the space of the quantum geometries. That is, a vector in K can be seen as a quantum linear superposition of different classical geometries, or a quantum superposition of Dirac operators. Each such states assigns not a fixed value, but rather a probability distribution over the possible values of the geometry defined by D. Accordingly, in the quantum theory the Dirac operator (an operator on H) is promoted to an operator on K, or, more precisely to an object which is at the same time an operator on H and on K:

$$\hat{D} = \begin{pmatrix} 0 & 0 & \hat{m}_1 \\ 0 & 0 & \hat{m}_2 \\ \hat{\overline{m}}_1 & \hat{\overline{m}}_2 & 0 \end{pmatrix}. \tag{33}$$

It is thus natural to consider the tensor product of H and K. We denote this tensor product as $\mathcal{H} = H \otimes K$. The vectors in $\mathcal{H}$ can be written as ψ^{ab}_n. In this basis, the matrix elements of the quantized Dirac operator $\hat{D}$ can be computed directly from (26) and (33). They are

$$\begin{aligned} \hat{D}^{abm}_{n\ cd} = \sqrt{\hbar G}\ \delta^b_d\ \big[&\sqrt{n}\ (\delta^a_1 + \delta^a_2)\ \delta^3_c\ \delta^{m-1}_n \\ &+ \sqrt{n+1}\ \delta^a_3\ \delta^{m+1}_n\ (\delta^1_c + \delta^2_c)\big]. \end{aligned} \tag{34}$$

Notice that this is a *single* operator, not a variable anymore (as the field operator in a quantum field theory is a single operator, not a variable as the classical field).

The algebra A, as well as J and χ, are still represented in $\mathcal{H}$ (they act trivially on K). Thus the quantum theory defines a novel triple $(A,\ \mathcal{H},\ \hat{D})$ with respect to which the set of the initial spectral triples (A, H, D) (for all possible D's), can be seen as a classical limit. The representation of (A, J, χ) in $\mathcal{H}$ is highly reducible. Each state ψ in K, namely each quantum geometry, defines a 3d subspace of $\mathcal{H}$ carrying a irreducible faithful representation π_ψ. We call D_ψ the operator with matrix elements

$$(D_\psi)^{ab}_{cd} = \overline{\psi}^n\, \hat{D}^{abm}_{n\ cd}\psi_m \tag{35}$$

acting on this subspace. Then the expectation value of the distance between any two states p and p' over A, in the quantum geometry determined by ψ is

$$d_\psi(p, p') = sup_{\{a \in A, |[D_\psi, \pi_\psi(a)]| \leq 1\}}\ |p(a) - p'(a)|, \tag{36}$$

where, notice, the Dirac operator and the representation of the algebra are restricted to the irreducible component. Conversely, each irreducible representation of (A, J, χ) in $\mathcal{H}$ determines a state in K. It is thus tempting to identify the quantum states of the geometry with these irreducible representations. We leave a more detailed analysis of the structure of the triple $(A, \mathcal{H}, \hat{D})$ for further developments.

VI. COMMENTS AND CONCLUSIONS

Spectral invariance. We have disregarded gauge invariance. The action we have chosen is not a spectral invariant: it is not a function solely of the eigenvalues of the Dirac operators, as is the GR spectral action (5). (The model we have chosen does not have enough degrees of freedom for accommodating gauges.) Spectral invariance is crucial in GR, as, in particular, it incorporates diffeomorphism invariance. The most interesting extension to the technique considered here should therefore, in our opinion, consists in the incorporation of spectral invariance. In the case of a spectral invariant action, the eigenvalues λ_n of the Dirac operators are natural real *gauge invariant* coordinates on $\mathcal{C}$. We expect that in the quantum theory they be represented as self-adjoint operators $\hat{\lambda}_n$. For GR, some information on the Poisson algebra of the λ_n's was obtained in [7].

Zero point distance. We may attach an intuitive physical interpretation to the "zero point distance" given by (31) with $n = 0$. In the classical theory the two points p and p' can be at infinite distance (when $m = 0$). In non commutative geometry, this can be viewed as a "classical" limit. In the quantum theory, on the other hand, there exist a maximal distance between p and p': the two "sheets" of the universe cannot separate in the quantum theory[††].

Dimensions. There are two natural possibilities of assigning physical dimensions to the Dirac operator in NCG. The traditional choice is to assign to the Dirac operator D the dimension of a mass. The second possibility is to assign it the dimension of an inverse length. If D is a mass, G has dimensions M/L. If D is an inverse length, G has dimensions $1/ML^3$. If D has the dimension of a mass, we must insert the Planck constant (or another constant with the same dimensions) in the formula (6) for the distance, in order to adjust the dimensions. For instance we can write

$$d(p, q) = sup_{\{a \in A, |[D,\pi(a)]| \leq \hbar\}} \ |p(a) - q(a)|. \tag{37}$$

This is not unreasonable, since in the commutative case the inequality is meant to fix the derivative of a to be less than one, but the (commutative) Dirac operator with dimensions of a mass is proportional to $\hbar \frac{\partial}{\partial x}$. On the other hand, it is a bit disturbing to invoke the Planck constant in a theory before quantizing it. Equation (37) yields, instead of (31)

$$d(p, p') = \frac{\hbar}{\sqrt{(2n+1)\hbar G}} = \frac{L_P}{\sqrt{2n+1}}, \tag{38}$$

where (32) is replaced by $L_P = \sqrt{\frac{\hbar}{G}}$. If, instead, D is an inverse length, all the equations in the text are dimensionally correct as they are. The two ways of assigning dimension are obviously equivalent. We find the second one (D is an inverse length), which does not require us to insert the Planck constant in the definition of distance, more natural from the perspective of a gravitational theory.

Functional quantization. An alternative approach to the canonical quantization method we have employed here is Feynman's sum over histories approach. To quantize our model a' la Feynman, we have to select a measure on the space of the histories, namely on $\mathcal{D} = C^2$ and integrate the exponential of the action with a source term added. This defines the generating functional, from which the Green functions of the theory can be computed by derivation. In a spectral invariant theory, this should yield an integration over the Dirac eigenvalues λ_n:

$$Z = \int [d\lambda_n] \ e^{-i/\hbar S[D]} \tag{39}$$

Details on the possibility and the difficulties of this approach will be given elsewhere.

In conclusion, we have studied the quantum theory of an elementary dynamical noncommutative geometry. We can derive a few lessons from this exercise. First, a key object is the space $\mathcal{D}_{(A,H)}$ of the Dirac operators compatible with given A and H. This is the kinematical arena of a dynamical noncommutative geometry. A second key object is the subspace Γ of $\mathcal{D}_{(A,H)}$ determined by the dynamical equations. An important problem is the determination of the symplectic structure of Γ. If a time evolution is defined, general techniques are available. These extend to the case in which time evolution is a gauge as in GR [7]. In the model studied here, Γ came equipped with a natural symplectic structure. In the quantum theory, the matrix elements of D are operators on a Hilbert space K. The two Hilbert spaces K and H combine naturally into the Hilbert space $\mathcal{H}$, on which a quantum Dirac operator $\hat{D}$ is defined (see (34)). The quantum states of the geometry are given by irreducible representations of the algebra in $\mathcal{H}$. The distance between two physical points (states over the algebra) which can take any nonnegative real value in the classical noncommutative geometry is quantized in the quantum theory, with a spectrum given in (1). The step from the simple model considered to a full theory including GR is obviously enormous. The results described form just an exploration of the structures involved in quantizing a non-commutative-geometry. Hopefully, these structures can be relevant in the quantization of more complete noncommutative models as well.

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